

# Entanglement of Identical Particles and the Detection Process

Malte Christopher Tichy,<sup>1</sup> Fernando de Melo,<sup>1,2</sup> Marek Kuś,<sup>3</sup> Florian Mintert,<sup>1</sup> and Andreas Buchleitner<sup>1</sup>

<sup>1</sup>*Physikalisches Institut, Albert-Ludwigs-Universität Freiburg,  
Hermann-Herder-Str. 3, D-79104 Freiburg, Germany*

<sup>2</sup>*Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven,  
Celestijnenlaan 200D, B-3001 Heverlee, Belgium*

<sup>3</sup>*Center for Theoretical Physics PAS, Al. Lotników 32/46, 02-668 Warszawa, Poland*

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We introduce *physical entanglement*, a unified entanglement concept for identical particles. We show that the particles' *effective indistinguishability* - controlled by the measurement setup - indicates whether the (anti-)symmetrization of the wave-function has any impact on physical observables.

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Entanglement manifests itself most counterintuitively in measurements at remote detectors which exhibit correlations stronger than allowed by realistic local theories [1]. For different types of particles, like electrons and protons, i.e. non-identical particles, entanglement can be defined rigorously [2]. Since the particles which carry the entangled degrees of freedom are intrinsically distinct, they have an undetachable, definite *identity*. The measurement of one particle can unambiguously be assigned to the density matrix acting on this particle's Hilbert space. The wave function of identical particles, however, has to be (anti)symmetrized for (fermions) bosons. Hence, the above identification of particles and Hilbert spaces breaks down. Instead, one must find other distinctive properties that allow to discriminate the particles. If two particles are localized far from each other, they define two distinguishable entities, related unambiguously to one local detector each. Between these entities, entanglement can be defined rigorously. The (anti)symmetrization procedure then does not affect any observable, and can thus be neglected.

Yet, in other settings, such clear differentiation is sometimes ambiguous or even impossible. Electrons that leave an ionizing helium atom, photons passing simultaneously through the input arms of a beam splitter in a Hong-Ou-Mandel setting (HOM) [6], or atoms in a BEC [7] are strongly overlapping in space, such that their external states do not allow to address the particles individually - they are effectively indistinguishable. Only the detection process finally reveals a subsystem structure which allows the definition of entanglement. Similarly, in the context of quantum correlations between different locations of the vacuum in quantum field theory [3, 4], the entanglement carriers are defined only at the very moment of the detection. Quite in general, the dynamical behavior of entanglement during transitions from overlapping, indistinguishable, to well localized, distinguishable particles is heretofore barely understood, while the interest in its role in atomic and molecular [8] as well as in biological systems [9] is growing.

In the present letter, we introduce a unified view on entanglement, which allows the consistent quantification of the entanglement between identical particles of variable degrees of distinguishability. We focus on the entanglement of particles rather than on the entanglement of modes [5, 14]. In previous approaches [11–13], the problem of indistinguishability is tackled by labelling the particles by some distinctive properties, e.g. their external state. We will refer to the latter as *a priori* entanglement [11–13, 15]. Much like distinguishable particles, two *a priori* non-entangled, *identical* particles will remain always unentangled unless interaction between them takes place. Their properties may change with time, however their identity, as given by their external states, will be preserved through unitary time evolution. By using *a priori* values, one makes the strong and crucial assumption that experimentalists are able and willing to always choose detectors that address distinct external properties of the particles. For separated particles, where each detector is assigned exactly one particle, this assumption is well justified.

In many situations, however, the external degrees of freedom and the detectors do not coincide anymore, and the *a priori* assumption is no longer justified. For example, in HOM-experiments, entanglement in the output arms of a beam splitter is created by the passage of two unentangled photons of opposite polarization through both input arms, and further post-selection on the coincident events [6, 16]. Only one-particle unitary evolutions occur, no interaction takes place between the photons, their wave-functions remain orthogonal throughout the whole process, while their spatial overlap changes dramatically. Therefore, entanglement measures like [11–13] still vanish, while entanglement is recorded by the detectors [6]. In view of these experiments, a characterization of entanglement that incorporates the measurement process is necessary to overcome such discrepancies.

For this very purpose, we introduce “physical entanglement”, that generalizes the previous approaches by incorporating the effects of the measurement process it-

self. In contrast to former treatments, we do not assign an absolute entanglement measure to a state on its own. Instead, we identify the particles as those entities which are eventually measured by some specified detectors. If each of the detectors is unambiguously related to exactly one of the particles, *physical entanglement* reduces to the one determined by the previously introduced criteria [12]. In contrast, an ambiguous detector setting, in which each particle can be measured by both detectors with a certain probability, can erase information about the provenience of the particles: the identification of the particles measured in the detectors with the particles prepared in the external states breaks down.

Our subsequent considerations apply for a general setting of arbitrarily many detectors, and particles with any internal degree of freedom, such as spin. For the sake of clarity, we restrict ourselves hereafter to the case of two particles and two detectors. In this context it is most convenient to formulate the problem in the suggestive language of first quantization. The measurement, and consequently the assignment of an identity to each particle, corresponds to take the expectation value of the operator

$$\hat{O}_d(\hat{\alpha}, \hat{\beta}) = \underbrace{\hat{O}_L \otimes \hat{\alpha}}_1 \otimes \underbrace{\hat{O}_R \otimes \hat{\beta}}_2 + \underbrace{\hat{O}_R \otimes \hat{\beta}}_1 \otimes \underbrace{\hat{O}_L \otimes \hat{\alpha}}_2 \quad (1)$$

where  $\hat{O}_L$  and  $\hat{O}_R$  are spatial projectors and describe detectors we call left and right, respectively.  $\hat{\alpha}$  and  $\hat{\beta}$  are observables on the internal spaces of the particles. The terms labeled by 1 (2) act on the external and internal spaces of the first (second) particle.

The two detectors assign an identity (left and right) to each of the detected particles, and therewith specify the entities which may or may not carry entanglement. The *physical entanglement* of a state  $\hat{\rho}_a$ , with respect to a given set of detectors, can thus be inferred by application of any entanglement measure on the detector-level density matrix  $\hat{\rho}_d$ , *reconstructed by quantum state tomography* [17] *in the basis of the detectors*. This corresponds to the measurement of a complete set of observables,  $\hat{\chi}_i, \hat{\chi}_j$ , on the internal degrees of freedom of the particles, with the two external detectors given by (1):

$$\hat{\rho}_d = N \sum_{i,j} \hat{\chi}_i \otimes \hat{\chi}_j \text{Tr} \left( \hat{O}_d(\hat{\chi}_i, \hat{\chi}_j) \hat{\rho}_a \right), \quad (2)$$

The normalization  $N$  ensures that  $\text{Tr}(\hat{\rho}_d) = 1$ . Thus, (2) describes the density matrix of the internal degrees of freedom as reconstructed by the detection procedure. In contrast to previous approaches, our formulation takes into account that the spatial overlap of identical particles and hence their distinguishability might change with time. The density matrix  $\rho_d$  obtained in (2) possesses a subsystem structure which reflects the the actual experimental setting. Therewith the problem of the indistinguishability of particles is overcome: Entanglement measures applied on  $\hat{\rho}_d$  determine the *physical entanglement*

of  $\hat{\rho}_a$  and reproduce the entanglement measured between the two detectors.

To access the continuous transition between distinguishable and indistinguishable particles, some definitions are necessary. The probability for two particles in the (not necessarily orthogonal) external quantum states  $|A\rangle$  and  $|B\rangle$  to trigger a coincident event is governed by the following quantities:

$$D_{LR} \equiv \langle A | \hat{O}_L | A \rangle \langle B | \hat{O}_R | B \rangle, \quad (3)$$

$$D_{RL} \equiv \langle A | \hat{O}_R | A \rangle \langle B | \hat{O}_L | B \rangle. \quad (4)$$

$D_{LR}$  ( $D_{RL}$ ) is the probability for the particle prepared in  $|A\rangle$  ( $|B\rangle$ ) to be detected in the left detector, while the particle prepared in  $|B\rangle$  ( $|A\rangle$ ) is detected in the right detector. If one of these quantities vanishes, a coincident event in the detectors reveals full information about which external state the particles were initially prepared in. Instead, if  $|A\rangle$  and  $|B\rangle$  are linearly dependent, the particles cannot be distinguished by any measurement acting on their external degrees of freedom, and the two probabilities must necessarily have the same value.

For the case that both  $D_{LR}$  and  $D_{RL}$  are finite, a click in one of the detectors does not completely reveal the provenience of the particles anymore. However, also in this case particles can still be effectively distinguishable, since one may discriminate the initial preparations by a further measurement: If  $\langle B | \hat{O}_{L/R}^\dagger \hat{O}_{L/R} | A \rangle = 0$ , the overlap of the projected states  $\hat{O}_{L/R} | A \rangle$  and  $\hat{O}_{L/R} | B \rangle$  vanishes, the states are orthogonal after detection, and can thus be distinguished. At the same time, expectation values for measurements on the internal degrees of freedom at the detectors are not affected by the (anti)symmetrization of the wave function. This procedure only plays a role if the overlap of the projected states does not vanish at both detectors. Hence a quantitative description for the impact of indistinguishability is given by the product of the overlaps at both detectors:

$$\gamma \equiv \langle A | \hat{O}_L | B \rangle \langle B | \hat{O}_R | A \rangle, \quad (5)$$

where we used that  $\hat{O}_{L/R}^\dagger \hat{O}_{L/R} = \hat{O}_{L/R}$ , for projector-valued  $\hat{O}_{L/R}$ . We baptize  $\gamma$  the *effective indistinguishability*. It includes, in contrast to previous approaches [18] to distinguishability, the dependence on the detector setting. The case  $\gamma = 0$  corresponds to the situation above in which particles are effectively distinguishable, and the information on the provenience of the particles is preserved during the detection process. For  $\gamma \neq 0$ , the (anti)symmetrization of the wave-function affects the calculation of expectation values. The absolute value of  $\gamma$  measures how strongly which-way information has been erased by the measurement setup: For  $|\gamma| = \gamma_{max} = \sqrt{D_{LR} D_{RL}}$ , the two states  $\hat{O}_{R/L} | A \rangle, \hat{O}_{R/L} | B \rangle$  are linearly dependent, and one cannot design any discriminating measurement. For particles initially in linearly independent external quantum states, such detection erased

the information about their provenience. They are hence fully indistinguishable for such detector setting.

For particles with some overlap at the detector, situations occur in which  $\gamma \neq 0$ , even if they are in principle distinguishable because of  $\langle A|B \rangle = 0$ . On the other hand, for any non-orthogonal states with  $0 < |\langle A|B \rangle| < 1$ , there are projectors which do differentiate the two states. Physical distinguishability hence strongly depends on the detectors' settings. For our purpose of exploring the consequences of indistinguishability on entanglement, it is sufficient to consider real and negative  $\gamma$  which corresponds to the case of HOM interferometry.

Let us now discuss the implications of the above for two identical particles prepared in external quantum states  $|A\rangle, |B\rangle$ . We assume that  $\langle A|B \rangle = 0$  and take their internal degree of freedom to be equivalent to a spin- $\frac{1}{2}$ -system. The following paradigmatic state will turn out to be strongly affected by ambiguous detector settings:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \{ \cos \epsilon |A, \uparrow\rangle_1 \otimes |B, \downarrow\rangle_2 + \sin \epsilon |A, \downarrow\rangle_1 \otimes |B, \uparrow\rangle_2 \\ + \delta (\cos \epsilon |B, \downarrow\rangle_1 \otimes |A, \uparrow\rangle_2 + \sin \epsilon |B, \uparrow\rangle_1 \otimes |A, \downarrow\rangle_2) \} \quad (6)$$

$\delta = 1(-1)$  refers to bosons (fermions). The subscripts denote the tensored Hilbert spaces of the two particles, and  $\epsilon$  is a real continuous parameter that controls the *a priori* entanglement between the particles. The entanglement measure concurrence [19] of (6) reads  $C_a(\epsilon) = 2|\cos \epsilon \sin \epsilon|$ . *A priori* maximally entangled (separable) states are obtained for  $\epsilon = \pi/4, 3\pi/4$  ( $\epsilon = 0, \pi/2$ ).

The specific choice of the *a priori* entangled state affects dramatically the coincident detection rate  $T$ . The latter is given by the expectation value of  $O_d(\mathbb{1}, \mathbb{1})$ , and reads  $T = D_{LR} + D_{RL} + 4\delta\gamma \cos \epsilon \sin \epsilon$ . The last term in this sum is proportional both to the *a priori* concurrence  $C_a(\epsilon)$ , and to the *effective indistinguishability* (5). Through the latter it includes the overlap of the single-particle states at the detectors, and the resulting (anti)bunching effects can be interpreted as a signature of exchange interaction. The case  $D_{LR} = D_{RL} = 1/4$ ,  $\gamma = -\gamma_{\max}$  corresponds to the situation in a HOM interferometer: Two photons in the  $|\Psi^+\rangle$  Bell-state ( $\epsilon = \pi/4$ ) always bunch ( $T = 0$ ), while in the  $|\Psi^-\rangle$  Bell-state ( $\epsilon = 3\pi/4$ ) they antibunch ( $T = 1$ ). This discriminates two maximally entangled states [20].

By virtue of (2), the state (6) yields the detector-level density matrix,

$$\rho_d = \frac{1}{2T} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T+a & b & 0 \\ 0 & b & T-a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (7)$$

in the detector level basis,  $|L \uparrow, R \uparrow\rangle, |L \uparrow, R \downarrow\rangle, |L \downarrow, R \uparrow\rangle, |L \downarrow, R \downarrow\rangle$ . Here,  $a = (D_{LR} - D_{RL}) \cos 2\epsilon$ , and  $b = 2\delta\gamma + 2(D_{LR} + D_{RL}) \sin \epsilon \cos \epsilon$ . For unambiguous detector settings,

$D_{LR} = 0, D_{RL} = 1$ , we simply recover the *a priori* state. In general, however, the state at detector level,  $\rho_d$ , is possibly mixed. We denote by  $S_d$  the two-particle linear entropy,  $S_d = 1 - \text{Tr}(\rho_d^2)$ , and by  $P_d$  the predictability [21] of the outcome of a measurement on a single particle. The former accounts for the impurity of the two-particle state, while the latter quantifies each party's ability to predict the outcome of a measurement in the  $\sigma_z$ -basis, and is given by

$$P_d^2 = |\text{Tr}(\sigma_z \rho_{red})|^2 = (D_{LR} - D_{RL})^2 \cos^2 2\epsilon / T^2, \quad (8)$$

with  $\rho_{red} = \text{Tr}_L(\rho_d)$  the reduced density matrix. Together with the physical concurrence  $C_d$ ,

$$C_d^2 = 4(\delta\gamma + (D_{LR} + D_{RL}) \sin \epsilon \cos \epsilon)^2 / T^2, \quad (9)$$

and for the state under consideration, the three quantities are complementary [22], by virtue of  $1 = C_d^2 + P_d^2 + 2S_d$ .

If  $\gamma = 0$ , the concurrence is unaffected ( $C_a = C_d$ ), ambiguous detector settings with  $D_{LR} \neq 0 \neq D_{RL}$  can manifest itself only as a higher two-particle entropy and lower predictability with respect to the unambiguous case. In this case, the identical particles behave distinguishably, hence just like particles of a different kind.

In contrast, non-vanishing values of  $\gamma$  change the situation dramatically, and the concurrence will be affected as well. For illustration, let us discuss two exemplary transitions, and their experimental implementations in HOM-type setups:

i) First, we consider a scenario with vanishing predictability,  $P_d = 0$ , by setting  $D_{LR} = D_{RL} = 1/4$ . The situation corresponds to a balanced HOM-type beam-splitter setup. In this case, neither party can predict at all any single spin measurement outcome. The nature of the single particle predictability depends on  $\gamma$ : For *a priori* non-entangled ( $\epsilon = 0$ ) and effectively distinguishable ( $\gamma = 0$ ) particles, the two-particle state (7) is maximally mixed, the uncertainty is purely classical,  $C_d = 0$ ,  $2S_d = 1$ . For effectively indistinguishable ( $\gamma = -\gamma_{\max}$ ) particles, the state at detector level  $\rho_d$  is pure, and the one-particle predictability stems totally from the *physical entanglement*. *By variation of  $\gamma$  the induced uncertainty is converted from classical impurity into physical entanglement!* The case  $\gamma = -\gamma_{\max}$  is analogous to the situation in HOM interferometry, in which pairs of maximally entangled particles are created [6, 20]. If, on the other hand, the temporal delay between the photons is too large such that the provenience of the particles could be in principle inferred by a time-resolved measurement (and  $\gamma = 0$ ), the particles result not to be entangled.

The smooth transition from distinguishable to indistinguishable particles shows an interesting behavior. For an initially separable state with  $\epsilon = 0$  or  $\epsilon = \frac{\pi}{2}$ , the concurrence increases monotonically with  $|\gamma|$ . The less the particles can effectively be distinguished, the more

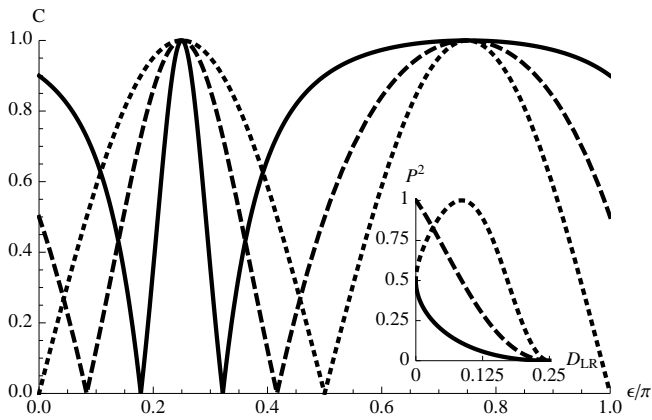


FIG. 1: Physical concurrence of two bosons in the state (6) as a function of  $\epsilon$  (which controls the state's *a priori* entanglement), for  $D_{LR} = D_{RL} = 1/4$ ,  $\gamma = 0$  (dotted line),  $\gamma = -0.5\gamma_{\max}$  (dashed line),  $\gamma = -0.9\gamma_{\max}$  (solid line). For  $\gamma = 0$ , the physical concurrence corresponds to the *a priori* concurrence. Inset: Squared physical single particle predictability  $P_d^2$ , as a function of  $D_{LR}$ , for  $\gamma = -\gamma_{\max}$ ,  $\epsilon = 3\pi/8$  (dotted line),  $\epsilon = 0$  (dashed line) and  $\epsilon = 5\pi/8$  (solid line).

they are entangled. In contrast, the behavior is non-monotonic and therewith more intricate for *a priori* entangled states, as illustrated in Fig. 1.

ii) As a second example, we now fix the *effective indistinguishability* at  $\gamma = -\gamma_{\max}$ , and tune the detector bias from unambiguous to completely ambiguous, by variation of  $D_{LR}$  between 0 and  $1/4$ , while  $D_{RL} = (1 - \sqrt{D_{LR}})^2$ . This transition can be implemented experimentally with several beam splitters with different transmission probabilities. Increasing  $D_{LR}$  corresponds to smoothly breaking the unambiguous bond between particle and detector with always maximal indistinguishability. The induced uncertainty is maximal for  $D_{LR} = D_{RL} = 1/4$ . No classical uncertainty is created by the detection,  $\rho_d$  remains pure. If  $D_{LR} \neq D_{RL}$ , the occurrence of a coincident event itself contains some statistical information on the possible provenience of the particles. The *physical entanglement* is then not maximal. While the measurement destroys the information about the preparation of the detected particles totally ( $\gamma = -\gamma_{\max}$ ), the probability for a certain type of coincident event (whether  $|A\rangle$  is measured in the left or right detector) to occur is biased. Again, predictability (and, due to the above-mentioned complementarity, concurrence) do not monotonically depend on  $D_{LR}$  (see inset of Fig. 1), with remarkable implications: when bunching occurs, the detector-level concurrence  $C_d$  can be smaller than the *a priori* concurrence  $C_a$ , the predictability (8) consequently grows. In other words, a spatial, spin-blind measurement can *enhance* the probability of a certain spin-measurement outcome. We illustrate this behavior in the inset of Fig. 1, where the squared physical predictability is plotted for different values of  $\epsilon$ , as a function of  $D_{LR}$ .

These examples illustrate how the effective indistinguishability  $\gamma$  provides a measure for the erasure of information on the particle preparation through the detection process, and hence for their effective indistinguishability. For vanishing  $\gamma$ , particles can be treated as distinguishable, since the (anti)symmetrization does not affect non-classical correlations. However, for  $\gamma \neq 0$  the (anti)symmetrization is crucial for expectation values and affects the entanglement between the measured particles. Our rigorous definition of *physical entanglement* incorporates these effects and is therewith suitable the experimental reality. While experiments hitherto mostly concentrated on highly engineered systems, and on entanglement between well defined entities, our approach allows to consider entanglement of initially indistinguishable particles in more natural contexts. In atomic or biological systems, for example, only the dynamics and the detection process eventually render the individual particles distinguishable [8, 9].

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